Prestress optimization of hybrid tensile structures

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Evolution of the structural weights
Evolution of the structural weights
Evolution of the structural weights

- Assyrians
- Romans
- Gothic
- Steel structures
- R.C. shells
- Suspended roofs
Evolution of the structural weights

\[ \frac{kN}{m^2} \]

\[ \text{wind depression} \]

\[ \text{year} \]

- Assyrians
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Evolution of the structural weights
Prestress optimization of hybrid tensile structures

Matrix analysis of pin-jointed frameworks

<table>
<thead>
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<th>Structure</th>
<th>Cables</th>
<th>Struts</th>
<th>Border elements</th>
<th>Prestress</th>
<th>Robustness</th>
</tr>
</thead>
<tbody>
<tr>
<td>TENSILE</td>
<td>✓</td>
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<td>✓</td>
<td>✓</td>
<td>✓</td>
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<tr>
<td>TENSEGRITIES</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>HYBRID TENSILE</td>
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- **Tensile Structures**: made of two set of cables, one set carrying the supporting function and the other exerting on the first a stabilization function;

- **Tensegrities Structures**: made of cables and struts internally self equilibrated through suitable prestress distributions and no requiring boundary containing stiffeners;

- **Hybrid Tensile Structures**: made of cable and struts, designed according to engineering intuition and based on many of the concepts of the first two typologies.
1. Matrix analysis of pin-jointed frameworks
   - An introductory example: the Von Mises Truss
   - The born of the mechanism and the way to contrast it
   - Equilibrium and compatibility equations

2. Example

3. Structural optimization by Genetic Algorithms
   - Formulation of the optimization problem
   - Results

4. Conclusions
1. Matrix analysis of pin-jointed frameworks
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An introductory example: the Von Mises Truss

Equilibrium \( \mathbf{At} = \mathbf{f} \)

\[
\frac{1}{l} \begin{bmatrix} a & -a \\ -h & -h \\ \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} P_x \\ P_y \end{bmatrix}
\]

Compatibility \( \mathbf{Bd} = \mathbf{e} \)

\[
\frac{1}{l} \begin{bmatrix} a & -h \\ -a & -h \end{bmatrix} \begin{bmatrix} d_x \\ d_y \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}
\]

Note: \( \mathbf{B}^T = \mathbf{A} \)

Constitutive law: \( \mathbf{t} = \begin{bmatrix} \frac{E_A}{l} & 0 & \frac{E_A}{l} \end{bmatrix} \mathbf{e} = \Phi \mathbf{e} \)

\( \mathbf{f} = \mathbf{A} \Phi \mathbf{B} \mathbf{d} = \mathbf{A} \Phi \mathbf{A}^T \mathbf{d} = \mathbf{K} \mathbf{d} \)
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Matrix analysis of pin-jointed frameworks

An introductory example: the Von Mises Truss

Equilibrium $\mathbf{A} \mathbf{t} = \mathbf{f}$
\[
\frac{1}{l} \begin{bmatrix}
  a & -a \\
  -h & -h
\end{bmatrix}
\begin{bmatrix}
  T_1 \\
  T_2
\end{bmatrix}
= 
\begin{bmatrix}
  P_x \\
  P_y
\end{bmatrix}
\]

Compatibility $\mathbf{Bd} = \mathbf{e}$
\[
\frac{1}{l} \begin{bmatrix}
  a & -h \\
  -a & -h
\end{bmatrix}
\begin{bmatrix}
  d_x \\
  d_y
\end{bmatrix}
= 
\begin{bmatrix}
  e_1 \\
  e_2
\end{bmatrix}
\]

Note: $\mathbf{B}^T = \mathbf{A}$

Constitutive law: $\mathbf{t} = \begin{bmatrix}
  \frac{EA}{l} & 0 \\
  0 & \frac{EA}{l}
\end{bmatrix} \mathbf{e} = \mathbf{\Phi e}$
\[
\mathbf{f} = \mathbf{A} \mathbf{\Phi B} \mathbf{d} = \mathbf{A} \mathbf{\Phi A}^T \mathbf{d} = \mathbf{Kd}
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An introductory example: the Von Mises Truss

**Equilibrium**

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**Compatibility**

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\frac{1}{l} \begin{bmatrix} a & -h \\ -a & -h \end{bmatrix} \begin{bmatrix} d_x \\ d_y \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}
\]

Note: \( B^T = A \)

**Constitutive law**

\[
t = \begin{bmatrix} \frac{E_A}{l} & 0 \\ 0 & \frac{E_A}{l} \end{bmatrix} e = \Phi e
\]

\[
f = A\Phi Bd = A\Phi A^T d = Kd
\]
Equilibrium $\mathbf{A} \mathbf{t} = \mathbf{f}$

$$\frac{1}{l} \begin{bmatrix} a & -a \\ -h & -h \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} P_x \\ P_y \end{bmatrix}$$

Compatibility $\mathbf{B} \mathbf{d} = \mathbf{e}$

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$$\mathbf{f} = \mathbf{A} \Phi \mathbf{B} \mathbf{d} = \mathbf{A} \Phi \mathbf{A}^T \mathbf{d} = \mathbf{K} \mathbf{d}$$
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Matrix analysis of pin-jointed frameworks

The born of the mechanism and the way to contrast it

Not aligned hinges

\[
\begin{bmatrix}
P_x \\
P_y
\end{bmatrix} = \frac{2EA}{l^3} \begin{bmatrix}
a^2 & 0 \\
0 & h^2
\end{bmatrix} \begin{bmatrix}
dx \\
dy
\end{bmatrix}
\]

\[
P_x = \frac{2EA}{l^3} a^2 dx \\
P_y = 0 + \frac{EA}{l^3} (dy)^3
\]

Aligned hinges

\[
P_x = \frac{2EA}{l^3} a^2 dx \\
P_y = \frac{2T_0}{l} (dy) + \frac{EA}{l^3} (dy)^3
\]

Aligned hinges, prestress
Prestress optimization of hybrid tensile structures

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Not aligned hinges

\[ \begin{bmatrix} P_x \\ P_y \end{bmatrix} = \frac{2EA}{l^3} \begin{bmatrix} a^2 & 0 \\ 0 & h^2 \end{bmatrix} \begin{bmatrix} d_x \\ d_y \end{bmatrix} \]

Aligned hinges

\[ P_x = \frac{2EA}{l^3} a^2 d_x \]
\[ P_y = 0 + \frac{EA}{l^3} (d_y)^3 \]

Aligned hinges, prestress

\[ P_x = \frac{2EA}{l^3} a^2 d_x \]
\[ P_y = \frac{2T_0}{l} (d_y) + \frac{EA}{l^3} (d_y)^3 \]
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Not aligned hinges

![Not aligned hinges diagram]

\[
\begin{bmatrix}
P_x \\
P_y
\end{bmatrix} = \frac{2EA}{l^3} \begin{bmatrix}
a^2 & 0 \\
0 & h^2
\end{bmatrix} \begin{bmatrix}
dx \\
dy
\end{bmatrix}
\]

Aligned hinges

![Aligned hinges diagram]

\[
P_x = \frac{2EA}{l^3} a^2 d_x \\
P_y = 0 + \frac{EA}{l^3} (dy)^3
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Not aligned hinges

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\begin{bmatrix}
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Aligned hinges

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P_x &= \frac{2EA}{l^3} a^2 d_x \\
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Aligned hinges, prestress

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\begin{align*}
P_x &= \frac{2EA}{l^3} a^2 d_x \\
P_y &= \frac{2T_0}{l} (d_y) + \frac{EA}{l^3} (d_y)^3
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P_x = \frac{2EA}{l^3} a^2 d_x \\
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Aligned hinges, prestress

\[
P_x = \frac{2E}{l^3} a^2 d_x \\
P_y = \frac{2T_0}{l} (d_y) + \frac{EA}{l^3} (d_y)^3
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**Prestress optimization of hybrid tensile structures**

**Matrix analysis of pin-jointed frameworks**

The born of the mechanism and the way to contrast it

### Not aligned hinges

\[
P_x = \frac{2EA}{l^3} a^2 \frac{d_x}{d_y}
\]

\[
P_y = 0 + \frac{EA}{l^3} (d_y)^3
\]

### Aligned hinges

\[
P_x = \frac{2EA}{l^3} a^2 \frac{d_x}{d_y}
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\[
P_y = \frac{2T_0}{l} (d_y) + \frac{EA}{l^3} (d_y)^3
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Aligned hinges, prestress

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Hypotesis

- members are connected by pin-joints;
- connectivity between the nodes and members is known;
- self-weight of members is neglected and the additional loads are applied only in nodes;
- linear elastic material.
Some definitions

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$j$</td>
<td>number of nodes</td>
</tr>
<tr>
<td>$b$</td>
<td>number of elements</td>
</tr>
<tr>
<td>$t$</td>
<td>forces in the elements $[b \times 1]$</td>
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<tr>
<td>$f_s$</td>
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<tr>
<td>$C_s$</td>
<td>connectivity matrix $[b \times j]$</td>
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$$
\Delta x = u = C_s x \quad \text{U} = \text{diag} \ (u) \\
\Delta y = v = C_s y \quad \text{V} = \text{diag} \ (v) \\
\Delta z = w = C_s z \quad \text{W} = \text{diag} \ (w) \\

\begin{align*}
\frac{x_i - x_h}{l_l} t_l + \frac{x_i - x_j}{l_m} t_m &= f_{ix} \\
\frac{y_i - y_h}{l_l} t_l + \frac{y_i - y_j}{l_m} t_m &= f_{iy} \\
\frac{z_i - z_h}{l_l} t_l + \frac{z_i - z_j}{l_m} t_m &= f_{iz}
\end{align*}

l_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}

\Rightarrow A_s t = f_s
### Some definitions

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\[
\begin{align*}
\Delta \mathbf{x} &= \mathbf{u} = \mathbf{C}_s \mathbf{x} \quad \mathbf{U} = \text{diag} \left( \mathbf{u} \right) \\
\Delta \mathbf{y} &= \mathbf{v} = \mathbf{C}_s \mathbf{y} \quad \mathbf{V} = \text{diag} \left( \mathbf{v} \right) \\
\Delta \mathbf{z} &= \mathbf{w} = \mathbf{C}_s \mathbf{z} \quad \mathbf{W} = \text{diag} \left( \mathbf{w} \right)
\end{align*}
\]

\[
\begin{align*}
\frac{x_i - x_h}{l_l} t_l + \frac{x_i - x_j}{l_m} t_m &= f_{ix} \\
\frac{y_i - y_h}{l_l} t_l + \frac{y_i - y_j}{l_m} t_m &= f_{iy} \\
\frac{z_i - z_h}{l_l} t_l + \frac{z_i - z_j}{l_m} t_m &= f_{iz}
\end{align*}
\]

\[
l_{ij} = \sqrt{\left( x_i - x_j \right)^2 + \left( y_i - y_j \right)^2 + \left( z_i - z_j \right)^2}
\]

\[
\begin{align*}
\mathbf{C}_s^T \mathbf{U} \mathbf{L}^{-1} \mathbf{t} &= \mathbf{f}_{x,s} \\
\mathbf{C}_s^T \mathbf{V} \mathbf{L}^{-1} \mathbf{t} &= \mathbf{f}_{y,s} \\
\mathbf{C}_s^T \mathbf{W} \mathbf{L}^{-1} \mathbf{t} &= \mathbf{f}_{z,s}
\end{align*}
\]

\[
\Rightarrow \mathbf{A}_s \mathbf{t} = \mathbf{f}_s
\]
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\Delta x &= u = C_s x \\
\Delta y &= v = C_s y \\
\Delta z &= w = C_s z
\end{align*}
\]

\[
\begin{align*}
U &= \text{diag}(u) \\
V &= \text{diag}(v) \\
W &= \text{diag}(w)
\end{align*}
\]

\[
C_s^T U L^{-1} t = f_{x,s} \\
C_s^T V L^{-1} t = f_{y,s} \\
C_s^T W L^{-1} t = f_{z,s}
\]

\[
\Rightarrow A_s t = f_s
\]
Matrix analysis of pin-jointed frameworks

Equilibrium and compatibility equations

\[
\begin{align*}
\frac{x_i - x_h}{l_l} t_l + \frac{x_i - x_j}{l_m} t_m &= f_x
\\
\frac{y_i - y_h}{l_l} t_l + \frac{y_i - y_j}{l_m} t_m &= f_y
\\
\frac{z_i - z_h}{l_l} t_l + \frac{z_i - z_j}{l_m} t_m &= f_z
\end{align*}
\]

\[
l_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}
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C_s^T UL^{-1} t = f_{x,s}
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\[
C_s^T VL^{-1} t = f_{y,s}
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C_s^T WL^{-1} t = f_{z,s}
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l_{ij} &= \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}
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<td>forces in the elements ([b \times 1])</td>
</tr>
<tr>
<td>(f_s)</td>
<td>nodal loads ([3j \times 1])</td>
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<td>(l)</td>
<td>length of the elements ([b \times 1])</td>
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<tr>
<td>(C_s)</td>
<td>connectivity matrix ([b \times j])</td>
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\[\Delta x = u = C_s x \quad U = \text{diag}(u)\]
\[\Delta y = v = C_s y \quad V = \text{diag}(v)\]
\[\Delta z = w = C_s z \quad W = \text{diag}(w)\]

\[
\begin{align*}
\mathbf{C}_s^T \mathbf{U} \mathbf{L}^{-1} \mathbf{t} &= \mathbf{f}_{x,s} \\
\mathbf{C}_s^T \mathbf{V} \mathbf{L}^{-1} \mathbf{t} &= \mathbf{f}_{y,s} \\
\mathbf{C}_s^T \mathbf{W} \mathbf{L}^{-1} \mathbf{t} &= \mathbf{f}_{z,s}
\end{align*}
\]

\[\Rightarrow \mathbf{A}_s \mathbf{t} = \mathbf{f}_s\]
Prestress optimization of hybrid tensile structures

Matrix analysis of pin-jointed frameworks

Equilibrium and compatibility equations

\[
\begin{align*}
x_i - x_h \frac{t_l}{l_m} + x_i - x_j \frac{t_m}{l_m} &= f_{ix} \\
y_i - y_h \frac{t_l}{l_m} + y_i - y_j \frac{t_m}{l_m} &= f_{iy} \\
z_i - z_h \frac{t_l}{l_m} + z_i - z_j \frac{t_m}{l_m} &= f_{iz}
\end{align*}
\]

\[l_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}\]

\[\Delta x = u = C_s x \quad U = \text{diag} (u)
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\]

\[
\begin{align*}
C_s^T U L^{-1} t &= f_{x,s} \\
C_s^T V L^{-1} t &= f_{y,s} \\
C_s^T W L^{-1} t &= f_{z,s}
\end{align*}
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\[\Rightarrow A_s t = f_s\]
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Equilibrium and compatibility equations

Equilibrium equations:
\[ A t = f \]

Compatibility equations:
\[ B d = e \]

Trought the *Virtual Work Principle* can be proved that:

\[ A = B^T \]

Through the exploration of the subspaces of the equilibrium matrix and of those of the compatibility matrix, it is possible to classify the pin-jointed framework (Pellegrino 1993). By defining:

- \( r_A \): rank of the equilibrium matrix;
- \( s = b - r_A \): state of self-stress (solution of \( A t = 0 \));
- \( m = 3j - k - r_A \): internal mechanisms (solution of \( B d = 0 \));

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The Singular Value Decomposition of the equilibrium matrix is used.
Equilibrium equations:  \( \mathbf{A} \mathbf{t} = \mathbf{f} \)

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The structure has an equilibrium matrix $A$ with $\text{rank}(A) = 35$:
- $m = 13$, number of internal mechanisms (no rigid motion);
- $s = 1$, number of self-stress state.
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- $m = 13$, number of internal mechanisms (no rigid motion);
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The structure in this arrangement can not be realized. Two types of engineering solutions may be adopted:

1. the introduction of additional bracing elements;
2. the introduction of prestress.

\[ m = 5, \text{ so 8 internal mechanisms are stablyzed.} \]

\[ m = 1, \text{ so 1 internal mechanism still remains.} \]
Example

1º Solution

\[ m = 0 \] and the final solution is a typical spatial truss, in which the bearing capacity is given by the *mechanical stiffness* of the assembly.
$m = 0$ and the final solution is a typical spatial truss, in which the bearing capacity is given by the mechanical stiffness of the assembly.
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$m = 0$ and the final solution is a typical spatial truss, in which the bearing capacity is given by the *mechanical stiffness* of the assembly.
The original structure has a single state of self-stress \((s = 1)\) (solution of \(At = 0\)).

It can be proved that this state is able to stabilize all the internal mechanisms, since the quadratic form \(\Lambda\) is positive definite.

<table>
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<tr>
<th>Groups</th>
<th>Prestress [N]</th>
</tr>
</thead>
</table>
| Cables 1-7 | \begin{tabular}{c|ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
Cables & 0.078 & 0.078 & 0.156 & 0.135 & 0.135 & 0.271 & 0.271 \\
Struts & -0.078 & -0.156 \\
\end{tabular} |

Now the stiffness of the dome is not due, like in the previous case, to a mechanical stiffness, but to a geometrical stiffness provided by prestress.
The original structure has a single state of self-stress \((s = 1)\) (solution of \(At = 0\)).

It can be proved that this state is able to stabilize all the internal mechanisms, since the quadratic form \(\Lambda\) is positive definite.

### Prestress \([N]\). Elements subdivided by groups (7 cables, 2 struts).

<table>
<thead>
<tr>
<th>Group</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cables 1</td>
<td>0.078</td>
<td>0.078</td>
<td>0.156</td>
<td>0.135</td>
<td>0.135</td>
<td>0.271</td>
<td>0.271</td>
</tr>
<tr>
<td>Cables 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cables 3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cables 4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cables 5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cables 6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cables 7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Struts 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Struts 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### 2° Solution

Now the stiffness of the dome is not due, like in the previous case, to a mechanical stiffness, but to a geometrical stiffness provided by prestress.
Additional requirements

- which is the effective value of prestress?
- what about the areas of the elements?
- which is the response of the structure under loads?
- is it safe? How much?
- is the deformability acceptable?

Charles Robert Darwin (1809-1882)
from On The Origins of Species

“In the struggle for survival, the fittest win out at the expense of their rivals because they succeed in adapting themselves best to their environment”
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Prestress optimization of hybrid tensile structures

Structural optimization by Genetic Algorithms

Formulation of the optimization problem

\[
\min_{x \in D} f(x), \quad \text{with } D = \{x \mid x^- \leq x \leq x^+, \ g(x) \leq 0, \ h(x) = 0\}
\]

Determine a vector \( x \in [x^-, x^+] \) that optimize the value of an objective function \( f(x) \), fulfilling a number of equalities \( h(x) = 0 \) and/or inequalities \( g(x) < 0 \) that represent the constraints of the problem.

Inspired by natural evolution, a genetic algorithm (GA) operates on a population of individuals (possible solution of the problem) by three basic operators:

- selection;
- crossover;
- mutation.

At the beginnin the population is randomly created. Then, throught the generations, the three basic operator create the new population.
Formulation of the optimization problem

$$\min_{x \in D} f(x), \text{ with } D = \{x | x^- \leq x \leq x^+, g(x) \leq 0, h(x) = 0\}$$

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Genetic algorithm for the minimization of a bi-dimensional function.
**Prestress optimization of hybrid tensile structures**

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---

### Loads

- Structural weight (P)

\[ d_{max} \leq \frac{l}{250} = \frac{18000}{250} = 72 \text{ mm} \]

- Live vertical load (Q)

\[ \gamma_s \cdot F_{Sd,t} \leq F_{Rd} \]

\[ \gamma_s \cdot F_{Sd,c} \leq \pi^2 \frac{EI}{l_o^2} \]

### Constraints

- Cables sections

\[ \Psi (\Psi \cdot t) \]

### Variables

- Struts sections 25:5:180

---

\[ \min_{x \in D} P(x), \ D = \{ x | \text{ respecting the constraints of the problem} \} \]
The genetic algorithm provided by modeFRONTIER is used.
1 Matrix analysis of pin-jointed frameworks
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Results

Prestress optimization makes us able to assign a prestress optimization. In particular it is shown how the hybrid tensile structures are spatial pin-jointed and how the constraint on deflection has a displacement during the construction of two cable domes for the Korean Construction of 2006. The example shows the application of the genetic algorithm, which is an optimization procedure allowing searching a solution of minimum cost. The optimal solution is shown in Table 5.

Table 5: Optimal solution.

<table>
<thead>
<tr>
<th>ID (C. Cable, S. Struts)</th>
<th>( \psi ) [N]</th>
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<td>2260000</td>
<td>7</td>
<td>9</td>
<td>18</td>
<td>13</td>
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<td>8</td>
<td>12</td>
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The optimal solution, obtained with Cable3, has been also studied with the commercial finite element software ABAQUS. This study aims to assess the fitness of the algorithm, which is an optimization procedure allowing searching a solution of minimum cost. The optimal solution is shown in Table 5.

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In order to stabilize the system, we implemented a hybrid prestressing procedure, which allows searching a solution of minimum weight. The resulted structure has now been stabilized by the additional bracing elements. Now, in alternative we can introduce two horizontal additional bracing elements. The optimal solution, obtained with Cable3, has been also studied with the commercial finite element software for form-finding problem of tensegrity structures.

The paper deals with cable domes design according to engineering intuitions. They are classified as stable. The succession of the generations.

Table 4: Prestressed cable dome. Prestressing system (initial force), final forces (due to dead and live loads), resistance, initial and final safety margins.

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<th>Cables 7</th>
<th>Struts 8</th>
<th>Struts 9</th>
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<tr>
<td>Initial forces [kN]</td>
<td>176.48</td>
<td>176.48</td>
<td>352.95</td>
<td>305.67</td>
<td>305.67</td>
<td>611.33</td>
<td>611.33</td>
<td>-176.48</td>
</tr>
<tr>
<td>Final forces [kN]</td>
<td>7.21</td>
<td>258.75</td>
<td>763.75</td>
<td>13.50</td>
<td>450.00</td>
<td>462.50</td>
<td>1325.00</td>
<td>-1045.00</td>
</tr>
<tr>
<td>Resistance [kN]</td>
<td>230.00</td>
<td>290.00</td>
<td>904.00</td>
<td>460.00</td>
<td>452.00</td>
<td>904.00</td>
<td>1327.00</td>
<td>-1060.78</td>
</tr>
<tr>
<td>Initial safety margins</td>
<td>1.30</td>
<td>1.64</td>
<td>2.56</td>
<td>1.50</td>
<td>1.48</td>
<td>1.48</td>
<td>2.17</td>
<td>6.01</td>
</tr>
<tr>
<td>Final safety margins</td>
<td>31.89</td>
<td>1.12</td>
<td>1.18</td>
<td>34.07</td>
<td>1.00</td>
<td>1.95</td>
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<td>1.02</td>
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- the solution of the optimization problem is achieved by a *genetic algorithm* specialized to optimal design of cable structures;
- the proposed algorithm has been applied to the structural optimization of a cable dome.

Stresa, July 2012

Thanks for your kind attention

quagliaroli@stru.polimi.it
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