Flexible bridge decks suspended by cable nets. A constrained form finding approach

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A R T I C L E   I N Fo

Article history:
Received 31 October 2012
Received in revised form 6 February 2013
Available online 4 April 2013

Keywords:
Cable nets
Suspended bridges
Form finding
Force Density Method
Extended Force Density Method

A B S T R A C T

The initial geometry of structures made of cables is steered by the cable tensioning forces. In a cable net the geometrical shape and the internal force distribution cannot be dealt as separate issues: the set of geometries defines also the feasible sets of the internal forces. During the last decades, many different approaches have been proposed to deal with the form finding of cable structures. The most efficient one is the so called Force Density Method (FDM), proposed by Schek, which allows to conforming cable nets for structural applications without requiring any further assumption, neither on the geometry, nor on the material properties. An Extension of the Force Density Method, the EFDM, makes it possible to set conditions in terms of fixed nodal reactions or, in other words, to fix the position of a certain number of nodes and, at the same time, to impose the intensity of the reaction forces. Through such an extension the EFDM enables us to deal with form finding problems of cable nets subjected to given constraints and in particular to treat mixed structures, made of cables and struts. In this paper we consider cable nets interacting with members having flexural behaviour. For a given cable assembly and for a given loading condition, aim of this work is to find that particular pretensioning system which replaces both the static and the kinematic functions of the inner reactions of a flexural elastic continuous beam. It is, for instance, the case of the bridge decks suspended by cables, shaped in various forms. The specialization of the EFDM to this type of problem is presented and a progressive set of examples shows the efficiency and the versatility of this approach in contributing to the design of new creative forms.

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1. Introduction

The initial geometry of the structures made of cables is steered by the cable tensioning forces. In a cable net the geometrical shape and the internal force distribution cannot be dealt as separate issues, as it happens in the case of the conventional structures: the set of geometries defines also the feasible sets of the internal forces. In the Sixties, when the first lightweight structures of this type were built, the only way to design cable nets was to resort to the use of physical models. The cable net form, the cutting pattern and the behaviour under external load were studied and measured through scale models and then assumed as basis for the design. In the same years the first rational solutions of the form finding problem were introduced. Barnes proposed a dynamical relaxation method (Barnes, 1975). Argyris developed a FEM approach suitable to deal with prestressed cable nets (Argyris et al., 1974). At the beginning of the Seventies, Linkwitz and Schek proposed the so-called Force Density Method (Linkwitz and Schek, 1971; Schek, 1974), that allows to conform cable nets for structural applications without requiring any further assumption, neither on the geometry, nor on the material properties. In its linear version, the final shape is defined through a special parametrization driven by the force densities. Schek presents also a non linear Force Density Method, that allows to deal with constraints concerning imposed relative distances between the nodes, the tensile level in the elements and/or their initial undeformed length. In this approach, the parametrization is not related to the nodes coordinates, but to each truss element. As Descamps et al. (2011) clearly observe, there is no direct control on free nodes coordinates. This is not a limit of the method, since it coherently assumes the nodes as free variable, otherwise the search of the form would be devoid of meaning. However, in dealing with systems in which it is necessary to fix the position of some additional nodes and at the same time to impose the value of the external force (as in the case of structures having a flexible beam/girder suspended to a cable net, or cable struts assemblies), many difficulties arise and the drawbacks of the FDM are self evident. In dealing with cable struts assemblies, Mollaert (1984) suggested an approach where the compression members are replaced by external forces. Tibert (1999) shows the possibility to overcome the drawbacks by introducing virtual elements in order to satisfy the specific requirements. The use of virtual elements is proposed also in Descamps
In dealing with lightweight bridge structures, in this last approach, the force densities in the virtual elements are gradually modified until they reach the fixed node location through an iterative procedure called Constrained Force Density Method. Another work that handles the geometrical position of some points of the net is the one proposed by Morterolle et al. (2012), that can be used to calculate geodesic tension trusses that ensure both contributions in the force density field come from Miki and Kawaguchi (2010), who reformulate the FDM in terms of functionals, on the basis of variational principle.

In this paper, the aforementioned problems are not solved by introducing virtual elements or virtual forces, but by proposing the missing relation between the force densities and the quantities related to the nodes. This is done through two steps: (1) computation of the reaction forces by using the same matrixes and vectors of the original work proposed by Schek; (2) writing the additional conditions in matricial form. This development can hence be applied in addition to the conditions posed by Schek (initial element lengths, final element lengths, element forces). The paper therefore extends coherently the operational capabilities of the original FDM, that becomes suitable not only to control the quantities related to the elements, but also the ones related to the nodes. The approach proposed can be used both in dealing with cable-struts systems as shown in Malerba et al. (2012) as well as in the case of structures having a flexible beam/girder suspended to a cable net. With this purpose, we consider cable nets interacting with members having flexural behaviour. It is the case of the long span bridges, whose deck girders are suspended at cables or supported by stays (Fig. 1). Whether using suspending cables or curtains of stays, the first design task concerns the setting of the initial configuration, which, for bridges, means the deck girder has to be horizontal or slightly cambered. Due to the interaction with the cables, a new development of the form finding problem is set. In the simplest view, cables or stays supply the static and the kinematic roles of the inner supports of a continuous beam (Fig. 1). The attainment of such a result requires a suitable pretensioning of the suspending system. The pretensioning of the cables is the means used to assign the initial configuration. In the case of stayed structures, for which the tension hardening behaviour of the suspending system is crucial, the pretensioning also provides the cables with the right stiffness and makes them able to play the static role assumed at the basis of these systems. In Sections 1 and 2 the FDM, in its linear and non linear forms, and the EFDM are recalled. Section 3 presents the specialization of the EFDM suitable to determine that particular pretensioning system which replaces the forces at the inner supports of the girder beam. A set of examples will show the efficiency and the accuracy of this approach in dealing with supporting cable curtains lying in a single plane, in two different planes or, generically, in the space. The same examples contribute to show the versatility of the method in helping the design of new original and creative forms.

2. An outline of the Force Density Method

We refer to a cable net and assume that:

- the net is made of straight cable elements, connected at the nodes. Part of the nodes is free, part of them is fixed;
- the net connectivity is known and its geometry is defined by the nodal coordinates;
- the cable elements are weightless;
- the net is subjected to concentrated forces, applied at the nodes.

The net has $n$ free nodes and $n_f$ fixed nodes, connected by $m$ elements. The total number of nodes is $n = n_f + n_i$.

With reference to the $i$th node of a 3D net (Fig. 2), the equilibrium equations in the $x,y,z$ directions are respectively:

$$L_i = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}.$$ (2)

2.1. Matrix formulation

In order to set out the equilibrium equations into a matrix form, the following vectors and matrices are introduced:

- $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, [n x 1], coordinates of the free nodes. By numbering the set of the fixed nodes after that of the free ones, the three vectors can be partitioned into the following subvectors:
  - $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, [n x 1], coordinates of the free nodes;
  - $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, [n x 1], coordinates of the fixed nodes;
- $\mathbf{f}$, $\mathbf{f}$, $\mathbf{f}$, [n x 1], nodal forces;
- $\mathbf{l}$, [m x 1], length of the elements; $\mathbf{L} = \text{diag}(\mathbf{l})$;
- $\mathbf{t}$, [m x 1], axial forces in the elements.

We define also a connectivity matrix $\mathbf{C}$, having dimensions $[m \times n]$, whose terms are:

$$c_i(e) = \begin{cases} +1 & \text{if } i = 1, \\ -1 & \text{if } i = 2, \\ 0 & \text{in the other cases}. \end{cases}$$ (3)

Fig. 1. Form finding of a cable suspending a flexible deck.

Fig. 2. Generical free node.
\[ u = C_{x}x; \]  
\[ v = C_{y}y; \]  
\[ w = C_{z}z. \] (4a)  

In this equation, by partitioning the matrix \( C \), we can put in evidence separately the coordinates of the free and those of the fixed nodes, as follows:

\[ u = C_{x}x + C_{f}x_{f}; \]  
\[ v = C_{y}y + C_{f}y_{f}; \]  
\[ w = C_{z}z + C_{f}z_{f}. \] (5a)  

By introducing the diagonal matrices \( U = \text{diag}(u), V = \text{diag}(v), W = \text{diag}(w) \) the free nodes equilibrium equations are expressed by the system:

\[
\begin{bmatrix}
    C_{UL}^{-1} & 0 & 0 \\
    0 & C_{VL}^{-1} & 0 \\
    0 & 0 & C_{WL}^{-1}
\end{bmatrix}
\begin{bmatrix}
    f_{x} \\
    f_{y} \\
    f_{z}
\end{bmatrix} = \begin{bmatrix}
    A_{t} = f
\end{bmatrix}
\]  

where \( A \) is the equilibrium matrix.

This system of equations can be used in two ways:

1. If the structural geometry is known, like happens in the usual structural design, by the approaches proposed by Pellegrino and Calladine (1986) and Pellegrino (1993) is possible to find all the static and kinematical properties of the assembly.

2. If we search for a particular structure, so that all the elements are tensioned, the system can be used as a form finding tooils. However, since the lengths of the elements depend on nodal coordinates, this system is non linear.

### 2.2. The linear Force Density Method

If we introduce the concept of force density \( q = T/L \):

\[ q = L^{-1}t. \] (7)

The equations of the system (6) became linear and uncoupled in the three cartesian directions:

\[ C_{x}^{1}Uq = f_{x}; \]  
\[ C_{y}^{1}Vq = f_{y}; \]  
\[ C_{z}^{1}Wq = f_{z}. \] (8a)

By introducing the diagonal matrix \( Q = \text{diag}(q) \), the following identities hold:

\[ Uq = Qu; \]  
\[ Vq = Qv; \]  
\[ Wq = Qw. \] (9a)

and Eq. (8) become:

\[ C_{x}^{1}Qu = f_{x}; \]  
\[ C_{y}^{1}Qv = f_{y}; \]  
\[ C_{z}^{1}Qw = f_{z}. \] (10a)

Substituting \( u, v, w \) as given by Eq. (5), we obtain:

\[ (C_{x}^{1}QC)x + (C_{y}^{1}QC)y = f_{x}; \]  
\[ (C_{y}^{1}QC)y + (C_{y}^{1}QC)y = f_{y}; \]  
\[ (C_{z}^{1}QC)z + (C_{y}^{1}QC)z = f_{z}. \] (11a)

and letting \( D = C^{1}QC \) and \( D_{f} = C^{1}QC_{f} \) we have finally:

\[ Dx = f_{x} - D_{x}x_{f}; \]  
\[ Dy = f_{y} - D_{y}y_{f}; \]  
\[ Dz = f_{z} - D_{z}z_{f}. \] (12a)

where solution is:

\[ x = D^{-1}(f_{x} - D_{x}x_{f}); \]  
\[ y = D^{-1}(f_{y} - D_{y}y_{f}); \]  
\[ z = D^{-1}(f_{z} - D_{z}z_{f}). \] (13a)

Given a net topology and assumed a vector \( q \) of force densities, the system (13) allows to find the unique equilibrium configuration of the system. Some examples are reported in Fig. 3.

### 2.3. Non linear Force Density Method

The linear formulation of the Force Density Method allows us to find all the possible equilibrium configurations of a net with a certain given connectivity and with given boundary conditions on the nodes. Each singular configuration corresponds to an assumed force density distribution. The possibility of imposing some further additional constraints should help us to find shapes not only equivalent, but also technologically sound. The possibility of imposing assigned relative distance among the nodes, the tensile level in the elements and their initial length, was once again introduced by Schek (1974).

If we suppose that all these conditions are function of the nodal coordinates and of the force densities, the generic additional condition assumes the following form:

\[ q_{i}(x, y, z, q) = 0 \quad (i = l, r; \ r < m) \] (14)

For all the \( r \) conditions introduced, we have:

\[ q(x, y, z, q) = 0. \] (15)

We choose an initial force density vector \( q^{0} \). For this assumed force density state, Eq. (15) is not in general satisfied. We search for a new vector:

\[ q^{(1)} = q^{(0)} + \Delta q \] (16)

so that \( q^{(1)} = 0 \). The solution is searched in an iterative form. We adopt the Newton method and searching for a vector \( \Delta q \) which satisfy the following linearized condition:

\[ g(q_{b}) + \frac{\partial g(q_{b})}{\partial q} \Delta q = 0. \] (17)

By calling

\[ G^{+} = \frac{\partial g(q_{b})}{\partial q} \] (18)

and

\[ r = -g(q_{b}). \] (19)

Eq. (17) becomes:

\[ G^{+} \Delta q = r. \] (20)

In this way we obtain a linear system, whose coefficient matrix has dimensions \( [r \times m] \). In a form finding problem, the number of the additional conditions \( r \) cannot be larger than of the number of the free parameters, which equals the number of the members of the net.

Being \( m > r \), the system (20) is underdetermined and admits \( \infty^{m-r} \) solutions. Among the infinite solutions we search that having minimum norm. In other words, among all the vectors which satisfy the system (20) we search the solution \( \Delta q \) which satisfy also the equation:
\[
\Delta q = \arg\min \| \Delta q \|_2^2
\]  
Eq. (20) and (21) form a problem of constrained optimisation, searching for the minimum of the function
\[
f(\Delta q) = \Delta q^T \Delta q
\]  
with the constraints
\[
G^T \Delta q = r.
\]  
We use the Lagrange multipliers method. We write Eq. (23) in the form:
\[
h(\Delta q) = G^T \Delta q - r = 0.
\]  
By introducing the Lagrangian function
\[
M(\Delta q, \lambda) = f(\Delta q) + \lambda^T h(\Delta q)
\]  
where \( \lambda \) is the new vector of the Lagrange multipliers, having dimension \( |r| \).

We search for stationary points of the lagrangian function. In order to determine such points, we make null the derivatives of this function with respect to \( \Delta q \) and \( \lambda \):
\[
\begin{aligned}
\frac{\partial M}{\partial \Delta q} &= 2 \Delta q + G \lambda = 0 \\
\frac{\partial M}{\partial \lambda} &= G^T \Delta q - r = 0.
\end{aligned}
\]  
In this way we obtain a square system of \( m + r \) equations, with \( \Delta q \) and \( \lambda \) unknowns. By combining the two Eq. (26), we have:
\[
\Delta q = G(G^T G)^{-1} r.
\]  
Being the initial conditions approximated through the linearization given by Eq. (17), the solution is reached in an iterative way. At the beginning of each iteration we assume:
\[
q^{(k+1)} := q^{(k)} + \Delta q^{(k)}.
\]  
Then, after the updating of the corresponding matrix \( G^T \) and of the vector \( r \) and, we compute through Eq. (27) the corrector of the force densities \( \Delta q \). The iterative process is stopped, when we obtain, with a given small tolerance:
\[
g(q^{(0)}) = -r(q^{(0)}) = 0.
\]  
2.3.1. A procedure to control the convergence
The convergence of the iterative method depends on the regularity of the function \( g \) and on the choice of the initial trial solution \( q^{(0)} \). If the convergence conditions are satisfied and, in particular, if \( q^{(0)} \) is sufficiently near to the solution, the Newton method converges with order 2:
\[
\| q^{(k+1)} - q \| \leq C \| q^{(0)} - q \| ^2.
\]  
For a given initial force density vector \( q^{(0)} \) the iterative solution technique may converge slowly or not converge at all. In order to control the convergence, we introduce a relaxed form of the Newton method. At each iteration, such a technique imposes a reduction of the norm of the residuals and so it is classified as norm reducing type. At each iteration \( k \) we pose:
\[
q^{(k+1)} = q^{(k)} + \alpha_k \Delta q^{(k)}
\]
where $0 \leq a_k \leq 1$ is a relaxing parameter, which, for instance, can be selected as follows (Quarteroni et al., 2007):

$$a_k = 2^{-(k-1)/2}, \quad k = 0, 1, \ldots, (32)$$

where $i$ is the first integer for which

$$||r^{(k-1)}|| \leq ||r^{(k)}||.$$ (33)

Usually $a_k$ starts with low values and, for the first iterations, close to zero. When the relaxed solution enters in the attraction zone of $q$, then $a_k$ tends to 1. For $i = 1$ and $a_k = 1$, this technique coincides with the original Newton method.

### 2.3.2. Jacobian matrix

The iterative solution involves an efficient formulation of the Jacobian matrix $G$. By adopting the chain rule derivation:

$$G^T = \frac{\partial g}{\partial q} = \frac{\partial g}{\partial x} \frac{\partial x}{\partial q} + \frac{\partial g}{\partial y} \frac{\partial y}{\partial q} + \frac{\partial g}{\partial z} \frac{\partial z}{\partial q} + \frac{\partial g}{\partial t} \frac{\partial t}{\partial q}$$ (34)

The derivatives $\partial x/\partial q, \partial y/\partial q, \partial z/\partial q$ are independent from Eq. (15) and can be expressed in terms of known quantities as follow:

$$\frac{\partial x}{\partial q} = -D^{-1}C^T U; \quad (35a)$$

$$\frac{\partial y}{\partial q} = -D^{-1}C^T V; \quad (35b)$$

$$\frac{\partial z}{\partial q} = -D^{-1}C^T W.$$ (35c)

Instead, the derivatives $\partial g/\partial x, \partial g/\partial y, \partial g/\partial z$ and $\partial g/\partial t$ depend on the assumed additional conditions (Eq. (15)). Explicit forms of these derivatives have been done to impose constraints on the distance between the end nodes, or on the forces acting in the elements or on the cutting lengths (Schek, 1974).

### 3. The Extended Force Density Method

#### 3.1. Introduction

As shown, the non linear Force Density Method allows to deal with constraints concerning imposed relative distances among the nodes, the tensile level in the elements and/or their initial undeformed length. Until now, no conditions have been set on the fixed nodes reactions. By introducing these new static parameters, new form finding conditions can be set. The possibility to impose conditions on the fixed end reactions, will allow us to solve new problems concerning structures made of nets and of other elastic elements, like bars and beams.

#### 3.2. Fixed end reaction computation

Eq. (1) sets the equilibrium equations of a generic free node of the net. The equilibrium of a generic fixed node is set in an analogous manner, by substituting the forces $F_i$ with the end reactions $R_i$, projected into their three components (Fig. 4).

![Fig. 4. Generical fixed node.](image)

In matrix form Eq. (36) becomes:

$$\begin{bmatrix}
C^{UL^{-1}} & R_f \\
C^{VL^{-1}} & R_v \\
C^{WL^{-1}} & R_w
\end{bmatrix}
\begin{bmatrix}
t \\
A_t
\end{bmatrix} = \begin{bmatrix}
R_e \\
R_v \\
R_w
\end{bmatrix}.$$ (37)

#### 3.3. Constraints on the end reactions

Through Eq. (37), which allows the end reaction computation, new form finding conditions can be set. The previous conditions were working on sets of $r$ elements. The constraints on the end reactions work on sets of the $n_f$ fixed nodes. We suppose that the constraints are set on a number $s \leq n_f$ of the fixed nodes. Each reaction has three components. We treat the reactions in each direction separately and compute the difference between the basic value of the reaction $R$ given by (37) and the value of the reactions that we want to impose $R_e$.

By writing the equations in matrix form, we have:

$$g_0 = R_e - R_{0f} = 0$$
$$g_r = R_{rf} - R_{rf} = 0$$
$$g_z = R_{zf} - R_{zf} = 0.$$ (38)

The vectors $R_{0f,rf}$ and $R_{zf}$ have dimensions $[s \times 1]$ and contain respectively the values of the end reactions and the prescribed values to be imposed. They are obtained by partitioning the vectors $R_{(x,y,z)}$, as follows:

$$R_e = C^{UL^{-1}}t$$
$$R_v = C^{VL^{-1}}t$$
$$R_w = C^{WL^{-1}}t.$$ (39)

Matrix $C^T$ has dimensions $[s \times m]$, as can be verified by the inspection of matrices and vectors present in Eq. 39. This matrix derives from matrix $C^T$, by extracting the row corresponding to the nodes to be constrained. It must be pointed out that, working on the nodes, and not on the elements, all the elements and all the terms of the matrices $U, V, W, L^{-1}$ and of the vector $t$ are involved in the computation.

#### 3.4. Jacobian matrix

With reference to Eq. (34), the derivatives of the nodal coordinates with respect to the force densities $\partial g/\partial q, \partial g/\partial q, \partial g/\partial q$ should be computed as before (Eq. (35)), while $\partial g/\partial x, \partial g/\partial y, \partial g/\partial z$ and $\partial g/\partial t$, depend on the new conditions to be imposed. We consider the vector $g$. The vectors $g_e, g_r, g_z$ should be treated in an analogous manner. Being $R_e$, a constant vector, we can write that:

$$\frac{\partial g_e}{\partial x} = \frac{\partial R_e}{\partial x}.$$ (40)

The dimensions of $R_e$ and $x$ and $\partial g_e/\partial x$ are respectively $[s \times 1], [n \times 1]$ and $[s \times n]$.

By deriving Eq. (40), we obtain:

$$\frac{\partial R_e}{\partial x} = C^{UL^{-1}}t$$ (41)
in which, both \( \mathbf{U} \) and \( \mathbf{L}^{-1} \) depend on \( \mathbf{x} \). Eq. (41) becomes easier to handle if we resort to the force density concept. By introducing the equation \( \mathbf{L}^{-1} \mathbf{t} = \mathbf{q} \) into Eq. (41) and remembering Eq. (9), we have:

\[
\frac{\partial \mathbf{R}_x}{\partial \mathbf{x}} = \mathbf{C}_l^T \frac{\partial}{\partial \mathbf{x}} (\mathbf{Uq}) = \mathbf{C}_l^T \mathbf{Q} \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \quad (42)
\]

From Eq. (5a) we obtain \( \partial \mathbf{u}/\partial \mathbf{x} = \mathbf{C} \) and Eq. (40) becomes:

\[
\frac{\partial \mathbf{g}_x}{\partial \mathbf{x}} = \mathbf{C}_l^T \mathbf{Q} \mathbf{C} \quad (43)
\]

The derivatives \( \partial \mathbf{g}_y/\partial \mathbf{y} \) and \( \partial \mathbf{g}_z/\partial \mathbf{z} \) are null, as can be seen in the following. Being \( \mathbf{u} = \mathbf{u}(\mathbf{x}) \) we have:

\[
\frac{\partial \mathbf{g}_x}{\partial \mathbf{y}} = \frac{\partial \mathbf{R}_x}{\partial \mathbf{y}} = \mathbf{C}_l^T \mathbf{Q} \frac{\partial \mathbf{u}}{\partial \mathbf{y}} = 0 \quad (44a)
\]

\[
\frac{\partial \mathbf{g}_x}{\partial \mathbf{z}} = \frac{\partial \mathbf{R}_x}{\partial \mathbf{z}} = \mathbf{C}_l^T \mathbf{Q} \frac{\partial \mathbf{u}}{\partial \mathbf{z}} = 0 \quad (44b)
\]

The equation giving \( \partial \mathbf{g}_x/\partial \mathbf{q} \) is obtained as follows:

\[
\frac{\partial \mathbf{g}_x}{\partial \mathbf{q}} = \frac{\partial \mathbf{R}_x}{\partial \mathbf{q}} = \mathbf{C}_l^T \frac{\partial}{\partial \mathbf{q}} (\mathbf{Uq}) = \mathbf{C}_l^T \mathbf{U} \frac{\partial \mathbf{q}}{\partial \mathbf{q}} = \mathbf{C}_l^T \mathbf{U} \quad (45)
\]

The Jacobian matrix has dimensions \( [s \times m] \) and is given by:

\[
\mathbf{G}_{xx} = \mathbf{C}_l^T \mathbf{U} - \mathbf{C}_l^T \mathbf{Q} \mathbf{C}^{-1} \mathbf{C} \mathbf{U} \quad (46a)
\]

\[
\mathbf{G}_{xy} = \mathbf{C}_l^T \mathbf{V} - \mathbf{C}_l^T \mathbf{Q} \mathbf{C}^{-1} \mathbf{C} \mathbf{V} \quad (46b)
\]

\[
\mathbf{G}_{xz} = \mathbf{C}_l^T \mathbf{W} - \mathbf{C}_l^T \mathbf{Q} \mathbf{C}^{-1} \mathbf{C} \mathbf{W} \quad (46c)
\]

with these equations we can solve the problem of finding the geometry of a net for which, in certain fixed nodes, the end reactions assume prescribed value in the three directions of the reference system.

3.5. Multiple constraints

We suppose to assign end reaction forces with arbitrary intensities and directions. This involves a generalization of the method, with the setting of multiple conditions. Let \( n_{sx} \) and \( n_{sy} \) the number of the constrained nodes respectively in \( x \) and \( y \) directions. By working with the Newton method, at each step the vector \( \Delta \mathbf{q} \) must satisfy both the conditions on \( x \) and \( y \), which are given by:

\[
\begin{cases}
\mathbf{G}_{sx} \Delta \mathbf{q} = \mathbf{r}_x = -\mathbf{g}_x \\
\mathbf{G}_{sy} \Delta \mathbf{q} = \mathbf{r}_y = -\mathbf{g}_y
\end{cases} \quad (47)
\]

or, in matrix form, by:

\[
\begin{bmatrix}
\mathbf{G}_{sx}^T \\
\mathbf{G}_{sy}^T
\end{bmatrix} \Delta \mathbf{q} = 
\begin{bmatrix}
\mathbf{r}_x \\
\mathbf{r}_y
\end{bmatrix}
\]

By letting:

\[
\mathbf{G}_k = 
\begin{bmatrix}
\mathbf{G}_{sx} \\
\mathbf{G}_{sy}
\end{bmatrix} \quad \mathbf{r}_{xy} = 
\begin{bmatrix}
\mathbf{r}_x \\
\mathbf{r}_y
\end{bmatrix}
\]

we have this final compact equation:

\[
\mathbf{G}_k \Delta \mathbf{q} = \mathbf{r}_{xy} \quad (50)
\]

Eq. (50) is analogous to Eq. (23). Only the dimensions of vectors and matrices changes: now the matrix \( \mathbf{G}_k \) and the vector \( \mathbf{r}_{xy} \) have respectively dimensions \( [(n_{sx} + n_{sy}) \times m] \) and \( [(n_{sx} + n_{sy}) \times 1] \), while \( \Delta \mathbf{q} \) maintains the dimension \( [m \times 1] \). From the computational point of view, it is sufficient to introduce and compile the set of reactions listed in Eq. (49). It can be observed that these new conditions can be used in addition to the original ones proposed by Schek.

4. Form finding for a mixed structures made of a flexural beam suspended by a cable net

4.1. A problem of design

In the design practice, the structures are conceived and dimensioned for certain dominant loads. After this stage, the structure is completely defined and it can be verified for the other loading conditions. In our case, a cable net is usually designed for the gravity and permanent loads. A first focus concerns the choice of its general layout, which, may be planar or spatial. Such a layout is predetermined according to different aesthetical, technical or economical criteria. The second choice concerns the actual form of the chosen layout.

Independently of the first choice, through the EFDM we can define the actual form of different proposable cable nets in such a way that they carry the elastic substructure with exactly the same suspending forces. Obviously, the different nets will have also different responses to the other loading conditions. These responses will be studied through some tool of analysis like, for instance, the FEM. In order to show the complementary role of these two tasks, let consider the continuous beam shown in Fig. 5. It can be supported, for instance, by:

(a) a single cable laying in the vertical plane;
(b) a single cable laying in an inclined plane;
(c) two cables laying in two inclined planes.

The beam is suspended at the cable net at the nodes 6–10. The nodes at the top of the antennas are numbered 11 and 12. The geometry of the suspension system should be defined through the nodes 1–5.

We search for that particular suspension system exerting at the internal supports 6–10 a set of forces which equals the fixed support reactions of the structure shown in (Fig. 6).

As shown before, the finding process works through the following steps:

- computation the reaction forces at the inner supports:

\[
\mathbf{X} = \frac{pl}{3T} \begin{bmatrix} 59 & 50 & 53 & 50 & 59 \end{bmatrix}, \quad (51)
\]

- application of the EFDM, by imposing the values of reaction in the \( z \) direction equal to those calculated (Eq. (51)). To obtain vertical hangers, the reactions in the \( x \) direction must be equal to zero;

- the compatibility of the system is set by fixing the nodes 6–10, Fig. 6;

- therefore the free nodes of the system are those numbered from 1 to 5.

4.2. The choice of the initial force densities

A typical question arising in form finding methods is the way the force densities are initially chosen. In fact, the solution of the constrained problem (23) is not unique and so the Newton method finds that more near to the initial distribution of force densities.

In order to guide the initial choice of force densities and to exclude non-consistent solutions, a specific attention to the static role
of the different groups of elements must be paid. A choice based on these assumptions is proposed:

- the main cable has a parabolic shape, defined through the ratio \( f/l \). The horizontal force \( H \) in the main cable is:

\[
H = \frac{pl^2}{8f} = \frac{pl}{8(f/l)}
\]

and the force densities in each cable is:

\[
q = \frac{H}{l/(n_h + 1)}.
\]

where \( n_h \) is the number of hangers.

- the hangers take the same load so that the force in each of them is:

\[
T_h = \frac{pl}{n_h}.
\]

The shape of the main cable is defined by:

\[
z(x) = 4f \left( \frac{x}{l} \right)^2 - \left( \frac{x}{l} \right)^4.
\]

from which we can evaluate the length of each hanger:

\[
l_h(x) = h - z(x).
\]

By considering: \( p = 15 \) kN/m, \( h = 10 \) m, \( l = 100 \) m, \( f/l = 1/12 \) and \( n_h = 5 \), the initial force densities are defined as follow: (See Table 1)

<table>
<thead>
<tr>
<th>No.</th>
<th>( x_h )</th>
<th>( z_p )</th>
<th>( l_h )</th>
<th>( q_{h0} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>16.67</td>
<td>4.63</td>
<td>5.37</td>
<td>55.86</td>
</tr>
<tr>
<td>2</td>
<td>33.33</td>
<td>7.41</td>
<td>2.59</td>
<td>115.71</td>
</tr>
<tr>
<td>3</td>
<td>50.00</td>
<td>8.33</td>
<td>1.67</td>
<td>180.00</td>
</tr>
<tr>
<td>4</td>
<td>66.67</td>
<td>7.41</td>
<td>2.59</td>
<td>115.71</td>
</tr>
<tr>
<td>5</td>
<td>83.33</td>
<td>4.63</td>
<td>5.37</td>
<td>55.86</td>
</tr>
</tbody>
</table>

For the main cable (elements 6–11) we have equal force densities:

\( q_{m,c} = 135 \) kN/m

The reaction forces to be imposed assume these values (Eq. (51)):

\[
X = [283.7 \ 240.4 \ 254.8 \ 240.4 \ 283.7] \text{kN}
\]

4.3. Results of the form finding process

By varying the position of the fixed nodes, through the EFDM we obtain the three different suspending systems shown, respectively, in Figs. 7–9. In the figures are reported also the geometries (in terms of free nodes coordinates) and the prestressing force in each element. The convergence curves for all the cases are reported in Fig. 10.

As shown in Figs. 11 and 12, the internal forces and the vertical displacements of the deck for the three cable nets are the same. This holds only under the load \( p \) considered in the form finding process. In fact, by considering an horizontal lateral load \( q = 1 \) kN/m, through geometrically non linear FE analyses we obtain the horizontal lateral displacements shown in Fig. 13.

5. Coping with complex geometries

The proposed method is able to solve the initial equilibrium problem for suspension systems with arbitrary three-dimensional shapes. In Fig. 17 three examples of cable nets supporting an elastic beam are shown. Now, we assume as free nodes all the internal nodes of the net. Concerning the choice of the initial force densities, we have assumed \( q = 1 \) in all the elements. In the first case, the suspension forces lie in the vertical plane only. In the second case, the goal of maintaining the same vertical supporting forces involves some horizontal bending actions in the deck. In the third case, by coupling two mirror suspending systems, these horizontal forces can be self balanced and the deck returns to work in shear/bending in the vertical plane only. The convergence curves for all the cases are reported in Fig. 16. As we can see from the diagrams of bending moments and shear forces (Fig. 14), the deck works once again as a beam on fixed supports. The achievement of this result, however, is accompanied by the raising of significant axial forces due to the horizontal components of the forces in the cables (Fig. 15).
6. Further considerations on the proposed examples

As clearly highlighted in the previous section, the paper is mainly addressed to propose a design tool in order to move from classical cable stayed-suspension configurations to new, more impressive morphologies. The proposed procedure is a constrained form finding approach that finds not only the particular prestressing system that satisfies specific static conditions, but also the structural geometry. Since, at this point, the geometry is known, we can use the system of Eq. (6) in order to get all the information about the scheme obtained with the above mentioned method.

Through the exploration of the subspaces of the equilibrium matrix it is possible to classify a pin-jointed framework, Table 2. In fact, by defining:

\[s = \frac{b}{c} - r_A, \quad m = \frac{3j}{c} - k - r_A\]  \( (57) \)

we have:

- \(s\): numbers of state of self-stress;
- \(m\): numbers of internal mechanisms;
- \(b\) is the number of elements;
- \(r_A\) is the rank of the equilibrium matrix,
- \(j\) is the number of nodes,
- \(k\) is the number of constraints.

As pointed out in Pellegrino (1993), all the information about the assembly can be obtained by the Singular Value Decomposition (SVD) of the equilibrium matrix. The states of self-stress are the solutions of \(At = 0\), the mechanisms are the solutions of the equations \(Bd = 0\). Matrix \(B\) is the compatibility matrix that is equal, for the virtual work principle, to \(A^T\).

If there are internal mechanisms, it must be verified that the obtained prestress state is able to stabilize all of them. This can be done according to Calladine and Pellegrino (1991), Calladine and Pellegrino (1992) and Zhang and Ohsaki (2006), by checking if the activation of generical mechanisms is possible or not.
Fig. 9. Case (c) two cables in two inclined planes.

Fig. 10. Convergence curves for the suspension bridges of Figs. 7–9.

Fig. 11. Shear force and bending moments in the supported beam (coincident for the three cases). Vertical load $p = 15$ kN/m.

Fig. 12. Vertical displacements of the supported beam (coincident for the three cases). Vertical load $p = 15$ kN/m.

<table>
<thead>
<tr>
<th>Node</th>
<th>$x$ [m]</th>
<th>$y$ [m]</th>
<th>$z$ [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>16.7</td>
<td>2.4</td>
<td>4.8</td>
</tr>
<tr>
<td>2</td>
<td>33.3</td>
<td>0.9</td>
<td>1.8</td>
</tr>
<tr>
<td>3</td>
<td>50.0</td>
<td>0.4</td>
<td>0.8</td>
</tr>
<tr>
<td>4</td>
<td>66.7</td>
<td>0.9</td>
<td>1.8</td>
</tr>
<tr>
<td>5</td>
<td>83.3</td>
<td>2.4</td>
<td>4.8</td>
</tr>
</tbody>
</table>

Hangers | Main cable

<table>
<thead>
<tr>
<th>No.</th>
<th>$F$ [kN]</th>
<th>No.</th>
<th>$F$ [kN]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>158.6</td>
<td>10</td>
<td>1095.7</td>
</tr>
<tr>
<td>2</td>
<td>134.4</td>
<td>6</td>
<td>1053.7</td>
</tr>
<tr>
<td>3</td>
<td>142.4</td>
<td>7</td>
<td>1035.9</td>
</tr>
<tr>
<td>4</td>
<td>134.4</td>
<td>8</td>
<td>1035.9</td>
</tr>
<tr>
<td>5</td>
<td>158.6</td>
<td>9</td>
<td>1053.7</td>
</tr>
<tr>
<td>11</td>
<td>1095.7</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
For example, for the cases shown in Figs. 7 and 8, the equilibrium matrix $A$ has dimension $15 \times 11$ and its rank is equal to 10. So there is a single state of self-stress ($s = 1$) and 5 independent internal mechanisms ($m = 5$), without rigid motions. The proposed cable nets are hence type IV assemblies. All the information can be obtained through the Singular Value Decomposition (SVD) of the equilibrium matrix and they are summarized in Figs. 18 and 19. It can be proved that the obtained state of prestress coupled with the obtained geometry provide a structure that is geometrically stable.

If we compare the self stress state obtained by SVD with the one obtained with the EFDM, we can observe that they are different. However, if we normalize the values listed in Fig. 7 and 8, we obtain the same values given by SVD (for instance: $283.7/2306.6 = 0.1230$ and $314.2/2331.7 = 0.1348$). In order to move from the results obtained by SVD to the ones obtained by EFDM, there is
a coefficient that includes the physics of the problem. Such a coefficient is given directly by the EFDM: for case (a), the coefficient is equal to 2306.6, while for case (b) it is equal to 2331.7.

We can consider also the more complex situation of Fig. 17(b). In this case, matrix $A$ has dimension $|135 \times 100|$ and his rank is equal to 99. Thus, there is a single state of self-stress ($s = 1$) and 36 independent internal mechanisms ($m = 36$), without rigid motions. The proposed cable net is a type IV assembly and it can be proved that the configuration is geometrically stable. One of the internal mechanisms is illustrated in Fig. 20. Once again, SVD gives

![Fig. 17. Three cable nets supporting an horizontal deck and hinged to: (a) a set of anchorages vertically aligned, (b) a set of anchorages laying on a swinging asymmetrical line and (c) a set of anchorages laying on a couple of mirror swinging lines.](image-url)
the same self stress state of EFDM, but with a different ratio between the values.

In conclusion, EFDM is able to find both the geometry and the actual prestress intensity in all the elements in order to satisfy specific statical conditions. The structure obtained through the EFDM can be further studied through a SVD of the equilibrium matrix, in order to catch more information about its static and kinematical internal characteristics. In cable systems we have to deal with a set of internal mechanisms. With reference to the case just presented we verified that to the prestressing state given by the EFDM, also stable configurations correspond.

Table 2
Classification of structural assemblies.

<table>
<thead>
<tr>
<th>Type</th>
<th>Static and kinematic properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>( s = 0, \ m = 0 )</td>
</tr>
<tr>
<td>II</td>
<td>( s = 0, \ m &gt; 0 )</td>
</tr>
<tr>
<td>III</td>
<td>( s &gt; 0, \ m = 0 )</td>
</tr>
<tr>
<td>IV</td>
<td>( s &gt; 0, \ m &gt; 0 )</td>
</tr>
</tbody>
</table>

Fig. 18. One of the five internal mechanisms of the suspension system of Fig. 7 and self stress state.

Fig. 19. One of the five internal mechanisms of the suspension system of Fig. 8 and self stress state.
7. Conclusions

A specialization of the Extended Force Density Method suitable to deal with cable nets supporting beams in flexure has been presented. For a given cable assembly and for a given loading condition, such a development allows to find that particular pretensioning system which replaces both the static and the kinematic functions of the inner reactions of a flexural elastic continuous beam. Due to the interaction with the cables, a new development of the form finding problem is set. The solution of this problem is of interest for long span bridges, whose suspended deck girders are to be maintained horizontal or slightly cambered through suitable pretensioning of the suspending cables or stays. Among other things, this method allows to deal with form finding problems in mixed systems, made of cables and struts, working in tension or in compression. A set of graduated examples shows the efficiency and the versatility of this approach in solving problems having complex geometries and/or constrains and in contributing to design new creative forms.

References


